

PRIMARY TEACHERS' MATHEMATICS CONTENT KNOWLEDGE: WHAT DOES IT LOOK LIKE IN THE CLASSROOM?

Peter Huckstep, Tim Rowland and Anne Thwaites

University of Cambridge

The mathematics subject matter knowledge of primary school teachers has in recent years become a high profile issue in the UK and beyond. There is statistical evidence (e.g. Rowland, Martyn, Barber and Heal, 2001) that secure knowledge of mathematics topics - including and extending beyond those encountered in the primary curriculum - is associated with more competent mathematics teaching in the case of pre-service ('trainee') primary teachers. Likewise, weak subject knowledge is associated with less competent teaching of the subject. This paper reports on a videotape study of 25 mathematics lessons prepared and conducted by trainee teachers. The aim was to identify ways in which their subject knowledge, or the lack of it, was evident in their teaching.

THEORETICAL FRAMEWORK

In Plato's famous illuminating example in *The Meno* (Plato 1956 edn) Socrates invites onlookers to witness his eliciting a classic mathematical misconception from a slave boy. The misconception concerned is the belief that the area of a square is proportional to the length of its side. Socrates draws the boy into the trap of supposing that this erroneous relationship holds before leading him out of it, ostensibly, solely by asking questions.

It is of special importance to note that Socrates' success in this exchange depends upon his own *knowledge* of mathematics. It depends, at the very least, upon his ability both to see the *implications* of the boy's responses and thus his readiness to *correct* such responses even if such correction is carried out obliquely by more questioning. Only someone with a certain level of knowledge could do this. The slave boy learns an important lesson for himself, but this learning depends on both Socrates' knowing a common misconception and also on his ability to introduce what we would now call 'cognitive conflict'. Thus, Socrates draws on a knowledge base that includes, but extends beyond, knowledge of mathematics *per se*.

The complexity and multiplicity of the knowledge bases required for teaching is now acknowledged internationally. In former times, however, it was supposed that in order to teach something, it was sufficient for the teacher to know it for him/herself. In particular, conceptions of mathematics teacher knowledge consisted of understanding what teachers knew about mathematics. This approach is not entirely without a rational basis. Aristotle wrote in antiquity: "... it is the sign of the man that knows, that he can teach..." (Barnes, 1984, p 1553). In the medieval universities, no distinction was made between knowledge of content and knowledge of how to teach it. Indeed, the modern form of doctoral examination (exposition and defence of the

thesis) originates in a medieval *inceptio* which was based on the belief that understanding was *demonstrated* by the act of teaching (Shulman, 1986). Dewey (1904) held that knowledge of subjects included knowledge of inquiry in the particular domain(s), and therefore knowledge of teaching method as he conceived it. More recently, John Wilson (1975) similarly held that comprehension of the logic of concepts offered guidance on how to teach them. Yet studies such as those of Begle (1968) and Eisenberg (1977) show that effective teaching in institutional settings requires more than personal mathematical competence.

By way of response, the seminal work of Lee Shulman conceptualises the diversity of the knowledge required for teaching. His seven categories of teacher knowledge include three with an explicit focus on ‘content’ knowledge: subject matter knowledge, pedagogical content knowledge and curricular knowledge. Subject matter knowledge (SMK) is knowledge of the content of the discipline *per se* (Shulman, 1986, p. 9) as represented, for example, by Bloom’s cognitive taxonomy. Shulman notes that the ways of discussing SMK will be different for different subject matter areas (*ibid.*), but adds to his generic account Schwab’s (1978) notions of substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community). For Shulman, pedagogical content knowledge (PCK) consists of “the ways of representing the subject which makes it comprehensible to others...[it] also includes an understanding of what makes the learning of specific topics easy or difficult ... (Shulman, 1986, p. 9). Curricular knowledge consists of knowledge of the scope and sequence of teaching programmes and the materials used in them. PCK is particularly difficult to define and characterise, but seems essentially to conceptualise the hitherto missing link between knowing something for oneself and being able to enable others to know it. Shulman explicitly includes knowledge of common misconceptions, and strategies for addressing them, as an important component of SMK.

In a study of the planning and instruction of novice secondary school teachers, Steinberg, Marks and Haymore (1985) found that teachers use various coping strategies when they lack content knowledge, including relying heavily on ‘the textbook’ as a convenient source of information. Lack of SMK also affected style of instruction, resulting in the avoidance of discussion and student questions. Notwithstanding the complex relationship between SMK and pedagogical content knowledge, there is evidence from the UK and beyond which would seem to support an increased emphasis on SMK (Ball, 1990a, b; Alexander, Rose and Woodhead, 1992; Ofsted, 1994; Ma, 1999). Ma, in particular, presents compelling evidence that the adequacy of elementary teachers’ substantive and syntactic knowledge of mathematics cannot by any means be taken for granted.

This evidential background lent support to an initiative of the UK government in specifying a curriculum for Initial Teacher Training (ITT) in England (DfEE, 1998), setting out what was deemed to be the “knowledge and understanding of mathematics

that trainees need in order to underpin effective teaching of mathematics at primary level”. The relevant government Circular (4/98) charged Initial Teacher Training ‘providers’ with the audit and remediation of students’ SMK. The Circular required providers to “audit trainees’ knowledge and understanding of [...] mathematics ...Where gaps in trainees’ subject knowledge are identified, providers of ITT must make arrangements to ensure that trainees gain that knowledge during the course” (DfEE, 1998, p. 48). There is now a growing body of research on prospective primary teachers’ mathematics subject knowledge, which has undeniably been facilitated by (‘inspired by’ might be overstating the case) this process of audit and remediation within ITT (e.g. Rowland, Martyn, Barber and Heal, 2000, 2001, 2002; Goulding and Suggate, 2001; Jones and Mooney, 2002; Sanders and Morris, 2000; Morris, 2001; Goulding, Rowland and Barber (forthcoming)). It should be noted that from September 2002, a new specification supersedes Circular 4/98, entitled *Qualifying to Teach: Professional Standards for the Award of Qualified Teacher Status* (TTA, 2002). These new requirements for ITT no longer specify a statutory body of mathematical subject knowledge, although guidance on the sources of evidence for “secure subject knowledge and understanding” is contained in a non-statutory handbook (para. 2.1, p. 10)

This paper is one outcome of ongoing collaborative work in this field between researchers at the universities of Cambridge, London, Durham and York (see also e.g. Goulding, 2002; Barber, Heal and Martyn, 2002) under the acronym *SKIMA* (subject knowledge in mathematics). The focus of the research reported in this paper is on ways that trainees’ mathematics content knowledge can be observed to ‘play out’ in practical teaching during school-based placements. The objective was to gain some insight into an earlier finding (Rowland *et al.*, 2001) that secure SMK, as assessed by the ITT mathematics audit, is associated with greater competence in both the planning and the ‘delivery’ of elementary mathematics teaching. Conversely, trainees with weak SMK are disproportionately likely to be assessed as weak in both teaching components. Whilst we know *that* this is the case, until now we have only been able to speculate as to *why* it might be so. An important dimension of this research was to identify ways in which trainees’ subject knowledge, or the lack of it, is evident in their teaching.

METHOD

This study took place in the context of a one-year (three term), full-time Post-Graduate Certificate in Education course for prospective primary school teachers in a university faculty of education. The primary ‘trainees’ are prepared to be generalist teachers of the whole primary curriculum. In this particular course, each of the 149 trainees followed a route focusing either on the ‘lower primary (LP)’ years (3-8) or the ‘upper primary (UP)’ (7-11). Trainees undertook a self-audit of their mathematical content knowledge in the first few weeks of the course, using materials provided in the course handbook. (This material and the actual audit instrument are common to the participating project universities, having been developed and trailed

collaboratively.) On the basis of this self-audit, each trainee made a report to their tutor, identifying any areas requiring further self-study (see Goulding, 2002). Trainees were referred to a number of books suitable for this purpose (such as Suggate, Davis and Goulding, 1998), and informal peer tutoring arrangements were put in place (see Barber, Heal and Martyn, 2002). About one month into the second term of the course, a 16-item audit instrument was administered to all the trainees, under semi-formal conditions. These audits were marked by tutors, who gave individual feedback to trainees indicating any further targets for self-study.

The trainees' response to each audit item was scored between 0 and 4 (using explicit and agreed criteria developed from trials). For the purpose of this research, the total scores for each paper (maximum 64) were used to identify groups with 'high', 'medium' and 'low' scores (above the upper quartile, between the upper and lower quartiles, below the lower quartile). Two trainees were then assigned to each cell of a 2x3 matrix, with rows corresponding to age-phase specialism (UP, LP) and columns to level of mathematics content knowledge as assessed by the audit (high, medium, low). As far as possible, the 12 trainees were chosen to reflect, as a group, the gender balance in the course as a whole. In practice, it was expedient that the 12 selected trainees had been placed in as few schools as possible for the final school-based placement. Ideally we aimed for pairs from 6 schools, and this constraint, in addition to the requirement of two trainees in each of the six cells, more or less determined the selection of the 12 who became the focus of the study. The usual ethical practices concerning consent and awareness of the purpose of their involvement were observed with respect to the trainees and the participating schools.

Two mathematics lessons taught by each of the trainees were observed and videotaped. These took place approximately in the 5th and 7th weeks of the 8-week placement; school half term occurred between the two observed lessons. In each case the observer/researcher was a PGCE course tutor (either one of the authors or one of two additional research officers). Trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson (usually the same day) the observer/researcher wrote a *Descriptive Synopsis* of the lesson. This was a brief (4-500 words) account of what happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These descriptive synopses were typically written from memory and field notes, with occasional reference to the videotape if necessary. In addition to straightforward description, two text styles were used to identify in the synopses (a) anything that the observer/researcher thought might turn out to be significant, or critical, moments or episodes with respect to the trainee's mathematics content knowledge, for consideration later by him/herself or another researcher (b) any evaluative comment within the descriptive synopsis; this was to allow occasional (in fact, quite rare) comments of the kind that one might write, as a *tutor*, on a lesson observation report (acclaim or criticism), yet which went beyond description. This was in part to alert others to the fact that the observer had imposed their own

evaluation onto the description, but also in recognition that such remarks might actually be found to have some analytical value later.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser and Strauss, 1987). A selection of the videotaped lessons was viewed by the whole project team (the three authors and two research officers) in extended, typically whole-day sessions. Additional tapes were viewed by two or three members of the team. At these meetings, we identified aspects of the trainee's behaviour that seemed to be significant in the limited sense that it could be construed to be informed by the trainee's mathematics content knowledge or their mathematical pedagogical knowledge. These were grounded in particular moments or episodes in the tapes. At first the identification of such moments, and accounts of their significance for our research, was in the form of proposals, or conjectures, for consideration by the team. They could be challenged or supported, and retained or rejected by consensus. We had to keep reminding ourselves that our research focus was *content* knowledge, and not other more general kinds of pedagogical awareness or expertise. For example, ensuring that all children are positioned to be able to see, say, a demonstration of counting by one of their peers is undeniably important, but not within the scope of our current enquiry. There were also times when we had to remind ourselves that we were not in the role of 'partnership tutors' (school placement supervisors). This inductive process generated a set of 18 codes, listed in the next session. Next, each of us viewed about five videotapes, and elaborated the *Descriptive Synopsis* into an *Analytical Account* of each lesson. In this account, significant moments and episodes were identified and coded, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature. The number of occurrences of each code in each analytical account were entered into a table.

Other members of the research team have re-visited and re-considered three of these codes in every analytical account. Nevertheless, we are aware that we are not well-placed with regard to inter-coder reliability at the time of writing this paper. We can, however, offer assurance that the accounts of the significance of the three codes selected for elaboration later in this paper have been considered and agreed within the team.

Lesson Structure

In the accounts of episodes within lessons which follow later in this paper, it is sometimes relevant to refer to the phase of the lesson in which the episode took place. This is made easier for us by the near-universal adoption of the lesson structure (as well as the curriculum) recommended in the National Numeracy Strategy *Framework* (DfES, 1999). This effectively segments every mathematics lesson into three distinctive and readily-identifiable phases. In the first, the *mental and oral starter*, the teacher works interactively with the whole class, typically (but not necessarily) on some well-defined mental calculation strategy. The second phase, the *main activity*, typically begins (*introduction* to the main activity) with some exposition by the

teacher of procedure or a key teaching point to the whole class, after which the class divides into *groups* (usually by attainment) to work on tasks set up by the teacher. In the final phase, the *plenary*, the teacher works again with the whole class to draw the threads of the main activity together, sometimes to offer extensions and different perspectives on it, and to remediate pupils' misconceptions.

OVERVIEW OF FINDINGS: THE CODES

As a result of the inductive process described above, 18 aspects of the trainees' teaching were identified, and a code assigned to each of them. We set out with the intention that each code would be firmly associated with subject matter knowledge (SMK), since this is the most obvious focus of the subject knowledge audit.

However, we abandoned this narrow focus early in our scrutiny of the videotapes, and agreed to include also several codes associated most readily with pedagogical content knowledge (PCK). Our reasons for doing so were first, that we found that the narrow focus on SMK had the effect of sidelining a great many interesting facets of the recorded teaching episodes. Secondly, we found ourselves agreeing with others (e.g. McNamara, 1991; Aubrey, 1997) who have argued that SMK and PCK cannot be separated into disjoint categories. This position is, of course, entirely consistent with that of Aristotle and others expressed earlier in this paper.

The 18 codes identified are as follows:

AC	anticipation of complexity	IE	identifying errors
AP	awareness of purpose	MC	making connections
ATB	adherence to textbook	OSK	overt subject knowledge
CE	choice of examples	RCA	recognition of conceptual appropriateness
COP	concentration on procedures	RCI	responding to children's ideas
CR	choice of representation	TU	theoretical underpinning
DA	deviation from agenda	UO	use of opportunities
DC	demonstration (child)	UT	use of terminology
DS	decisions about sequencing		
DT	demonstration (teacher)		

The code name should, in most cases, indicate the aspect of teaching intended to be associated with it. In principle, an event or episode in a lesson might be identified and coded as a *positive* or a *negative* instance of the any of the above aspects. Thus CE+, CE- would indicate respectively a good or poor choice of example(s). For other codes the existence of both possibilities is less apparent. It is not easy, for example, to envisage a positive instance of concentration on procedures (at the expense of conceptual learning), but we were open to that possibility. It is not our intention, here, to clarify the meaning of each of each of the codes. Three of them - MC, RCI, and CE have been selected for elaboration in the remainder of this paper. We note, with some surprise, that none of the 18 codes refers explicitly to *questioning* or *explaining*, both of which are recognised to be key pedagogic competencies (e.g. Wragg and Brown, 1993; Brown and Wragg, 1993). In fact there were plenty of instances of trainees

asking questions, but very few stood out as significant, for good or ill. Socratic dialogue was not in evidence, whilst there were many examples of “teacherly questioning” (Gadamer, 1990) such as “What did we do yesterday?” or “What do we call this shape?”. Explaining, on the other hand, is inherent in some of the codes identified and listed above.

MAKING CONNECTIONS

The pursuit of mathematical *connections* in teaching pedagogy has intensified in recent years. Askew *et al*, for example, have singled out teachers with a so-called “connectionist orientation” as those who are more likely to be effective teachers of numeracy than those with certain other beliefs about teaching and the nature of mathematics (Askew, Brown, Rhodes, Wiliam and Johnson, 1997). For Askew and his colleagues a “connectionist” is the holder of a cluster of beliefs. More succinct accounts can be found in the work of earlier writers who have also strenuously argued for this vital element of teaching. Ball (1990b), in particular, outlines essential subject knowledge in which she includes the imperative that “teachers must appreciate and understand the connections among mathematical ideas”, more explicitly that they should understand:

... how fractions are related to division, how place value figures in multiplication computation, and the connections and distinctions among measurements of distance, area, and volume (Ball, 1990b, p. 458)

Of course, this is not an exhaustive list of the source of possible connections. Haylock (1982) for example, has provided a network of connections that included links between mathematics and extra-mathematical situations, as a helpful means of identifying pupils’ understanding. Rich mathematical connections - just like the selection of good examples - can be determined *pre-actively* in the planning stage of teaching. However, often as a result of responding to children’s ideas, a suitably mathematically informed teacher can also take advantage of opportunities to make connections *interactively*. In our study notable examples of both pre-active and interactive connection-making were evident.

The Trainees’ Connection-Making

A fairly obvious missed opportunity for making connections that could have been drawn out in the planning stage, is provided by Jason. In the main teaching section of his lesson with a Year 3 class he seems to have wanted to draw out from the pupils a *definition* of fractions. He began his enquiry by, rather disingenuously, feigning ignorance:

- Jason: What’s a fraction? I don’t know, tell me what a fraction is ...
Gary: It’s a number, a line and then another number.
Jason: So ...
Gary: But the top number has to be smaller than the bottom number.
Jason: Yes, you’ve described a fraction. ... but what is a fraction? Elliot.

Clearly, this response did not satisfy Jason. Gary had given a perfectly general account of fraction *notation* but for Jason this, in a more fundamental sense, is not what a fraction *is*. His approval of Elliot's rather incomplete response: ("it's an equal amount of something") suggests that he wanted to define fractions in terms of equal parts and he continued to develop this idea using *regions* of shapes.

There is evidence in the literature to suggest that this model should take temporal pedagogical priority over the fractions-of-a-number model (e.g. Dickson, Brown, and Gibson, 1984). Yet, in an earlier section of his lesson, Jason had already played a game in which children were finding the fractions of a given number in order to earn a point for their team. They had found tenths, quarters, halves and fifths of the numbers that he provided, and although one child did query the meaning of 'quarter', the question was still answered correctly. Furthermore, since all of the responses provided were correct, and Jason had given no instruction, we can infer that this work had been covered before. Of course, it may not have been made explicit to the children that they had been finding *fractions*, since they might have just been given a procedure for answering questions of the form what is a ... of...? Nevertheless, this work could profitably have been linked to the work of the main section which, as we have seen, dealt head-on with the meaning of a fraction. So, given that the children had already some competence at finding the size of each equal part of a selection of numbers, we must conclude that links to this earlier work were either overlooked in the planning stage, or were not recognised as being grist to the fraction-concept mill. We can say, therefore, with some justification, that Jason's subject knowledge is in this respect deficient.

Caroline, on the other hand, displayed ingenuity in precisely the same context with a Year 2 class. Unlike Jason, Caroline did inform her class that the work in the introductory section (reciting the two times table) would be important for the later activity. More importantly, she attempted to make a crucial link between fractions of shapes and fractions of numbers, even though an unfortunate, though not irredeemable, flaw remained in the very promising resource that she had devised.

In her introduction to the main activity phase, having carried out some divisions of regular shapes into congruent regions, she placed on the board a circular card with four smaller circles attached at equal distances round the circumference. It was supposed to represent a flower. She encouraged the children to see how halving this configuration determines half of the number of petals. Of course, if this quotient is to be displayed in whole petals it will depend upon where the halving line is placed. Clearly, if the line passes through a petal then the petals themselves would be partitioned making the model less straightforwardly instructive. However, *she* positioned the line each time to ensure that this complication was avoided.

Unfortunately, for her next example Caroline produced an eight-legged spider configuration any two legs of which were thin enough to be completely hidden by a 'degenerate' line represented by the broad strip of card that she was using for her dissections. This did not go unnoticed by one child who demonstrated that by placing

the strip through a pair of legs, only three legs remained in each half! Caroline was not sympathetic to this response. She simply placed the dividing line elsewhere without acknowledging that the child had found a loophole in her model.

Despite the hiccup, this episode of Caroline's lesson remained a bold attempt at connection-making and, moreover, she went on to make further impressive mathematical connections *interactively*. From the spatial models, she went on to explore ways of finding a half and a quarter of various numbers and was surprised and delighted by one child's method of recognising that a half is one part of a known double. She linked this idea to addition, multiplication and sharing and also she drew out the role of the denominator in finding fractions of numbers. In this way she formed a network of approaches to finding fractions before setting the class independent tasks to work at. Both her pre-active and her interactive links were informed by non-trivial subject knowledge.

In an otherwise fine lesson with a Reception class, Colin exemplified a failure to make a subtler connection within his lesson. During the introductory activity he asked one child, Andrew, to demonstrate publicly the number of elephants shown on a card by counting. Colin commended Andrew on his counting strategy which he drew to the attention of the class by asking them "How did he count [then] so that he didn't count any of them twice?" In order to emphasise Andrew's systematic approach, he reflects back to the class their response that Andrew went "down and down" the columns.

In the introduction to the main activity, large magnetic coins had been stuck onto a white board. Colin produced a toy dog bearing a 5p label and the children were asked to discuss in pairs the combinations of coins that could be used to buy this toy. It was Andrew, again, who selected a 2p and three 1p coins. However, in verifying this combination publicly Colin reached the required price of 5p by counting the two pence coin *twice* since he was now counting the *value* of this coin. But this important shift – one with which many young children often have considerable difficulty - was passed over by Colin without comment. Given his earlier praise of Andrew, for counting in such a way that one object was not counted twice, Colin's counting here would appear to need justifying. He repeated this in his public checking of Samantha's choice of coins which are two 2p coins and one 1p coin. In not anticipating the distinction between counting *natural* objects and *conventional* value Colin displays a shortfall in his knowledge which is highly pertinent in successful teaching in this context.

For each of these three trainees, we maintain that the decision to either embody (or overlook) precise links in their teaching reflects the strength (or weakness) of certain knowledge - either PCK or SMK.

RESPONDING TO CHILDREN'S IDEAS

A detailed philosophical account of teaching has been provided by Passmore (1980) who reminds us that teaching is, amongst other things, a *triadic* relation. The teacher,

child and subject all have a part to play in the teaching and learning process; as Passmore put it, “For all X, if X teaches, there must exist someone who, and something that, is taught by X” (p. 22). This relationship between the teacher and the child can be of great importance to the child’s learning and we shall see how the teacher can use the ideas that a child offers to enhance their learning of the subject.

A constructivist view of learning provides one fundamental perspective on children’s contributions within lessons. When a child articulates an idea, this points to the nature of *their* knowledge construction, which may or may not be quite what the teacher intended or anticipated. These articulations may be elicited by a teacher’s questions, or volunteered unsolicited, as the examples given below illustrate. To put aside such ideas, or simply to ignore them or dismiss them as ‘wrong’, is an expression of lack of interest in what it is that that child (and possibly others) have come to know as a consequence, in part, of the teacher’s teaching. In short, unless a teacher responds appropriately to children’s ideas, they cannot truly be said to be teaching.

The way in which the teacher and child interact in the classroom was highlighted by the Cockcroft Report (DES, 1982) which emphasised dialogue and discussion within the teaching of mathematics (see, for example, paragraph 243). Brissenden (1985) believes that children should not be “... inhibited by what they feel to be the teacher’s judgement of them” but that they should be learning in “an atmosphere which encourages both the tentative offering and the development of more formal discourse when this becomes important”. He therefore argues for the need to reconsider the ways in which teachers act upon children’s responses during lessons. In particular he urges that, as a teacher, “... you should resolve NEVER to evaluate your children’s responses” (emphasis in the original). A key element in breaking the classic Initiate-Response-Feedback cycle is the necessity for the teacher to listen “... carefully to what children actually say and mean” (p. 6), and then to respond to it.

The Trainees’ Response to Children’s Ideas

The videotaped lessons gave rise to a large number of codes in this category, both positive and negative, and these can be broken down into three types. The first type is the *child’s response to a question* from the teacher (the trainee, in this case); the second is a *child’s response to an activity or discussion*; the third is *when a child gives an incorrect answer* to a question or in the course of a discussion.

In the first two of these the unexpectedness of the child’s answer, comment or observation is critical and the trainee’s own grasp of subject content knowledge is one of the factors determining the way that they deal with it. We now give examples of these first two situations.

During the introduction to the main activity of Colin’s lesson with the reception class described earlier, when the children were asked which coins could be used to pay for a toy dog priced at 5p. After Andrew and Samantha’s answers (2p, 1p, 1p, 1p and 2p, 2p, 1p), Colin requested one more way, presumably expecting 1p, 1p, 1p, 1p, 1p. An interesting, unexpected response was made by Niall, who suggested 10p – 5p.

Although this was not the response that Colin had expected, he welcomed it and shared it with the class. He was broadening the possible combinations to include subtraction as well as the intended addition and deviated, for a few moments, from his lesson plan. He could, of course, have gone further and related it to the idea of change but he did not do so. Here it seems that the child had construed the situation in a way that the teacher had not but which the teacher heard and embraced. In responding to the child he was allowing the child to make the types of connections that have been discussed in the previous section.

Another example of a trainee asking a question and receiving an unexpected response arises from Caroline's lesson on fractions with a Year 2 class. Again, other aspects of this lesson have been highlighted in the previous section. In her introduction to the main activity, she asked the children to "split a shape in half". First, they considered a paper square which children folded along the four line of symmetry. Fixing the square on the whiteboard, she then placed paper strips on the two axes parallel to the sides, thus dividing the square first into halves, then into quarters. Next they considered an oblong which Caroline had fixed to the board, intending that it be split similarly into halves and then quarters. However, one child, Simon, wanted to mark in the diagonal in order to split the oblong in half. The way in which Caroline had posed the question, "How would you split that in half?", had given a broader range of possibilities than she had anticipated – this can be inferred from the way she summarised the activity. Whilst Caroline did not want to explore halving this way, she asked the class whether such a line would split the shape in half. She suggested that we "imagine we turn this round the other way". In saying this she showed her ability to respond well to the child's idea and also realised that the matter could be settled by performing a rotation, whereas reflection had been implicit in the earlier folding activity.

The next shape she considered was a circle and this time she did not respond at all to a child who called out three times "You can do it anywhere". The child was clearly wanting to point out that any diameter would 'split' the circle into halves. Caroline ignored this response and drew in one line of her own choice, and then a second at right angles to show quarters. It is not clear whether Caroline was unwilling to deviate from her agenda at this stage of her introduction, or whether she was unaware of the mathematical significance, for the whole class, of the child's observation.

Sometimes an unexpected response during the on-going development of a lesson can be quite worrying to the teacher. One strategy observed to be adopted by trainees in such circumstances is to deflect discussion away from the unexpected contribution, thus making space for the trainee to think about their response to it. An example of this arose during the (lengthy) introduction to the main activity of Laura's second videotaped lesson with a Year 5 class. She had introduced an investigation of making symmetrical shapes using two identical U-shapes. She drew one of them and explained that this was only half of a completed shape; then she drew the other half in one possible position. She asked the children, "What could you say?" about the

completed shape. Several suggested that it was symmetrical and the children showed the lines of symmetry. Then she drew another possibility. Laura returned to Tom who had offered an idea earlier and the child commented that both of the completed shapes formed a letter of the alphabet. Laura admitted that he had found a “pattern” that she had not seen.

Laura: Tom, you’d noticed a pattern there which I hadn’t. You said that they looked like letters.

Jamie: Yes.

Laura: They do, what letters do they look like?

Jamie: They look like an H and an O.

Laura: OK, a capital H and a capital O. That’s good. It’s always good to be reasoning and thinking beyond what we’re doing, but that’s not actually what we’re, I didn’t realise that they were going to make letters, so you noticed a pattern that I hadn’t even thought about. OK, but that doesn’t have to be the way it is. You don’t have to confine yourself to that.

Over six minutes later, the following interchange occurred following another two examples having been drawn on the large sheet of squared paper:

Laura: What are all these hands up for? Is it another letter?

Children: Yes.

Laura: OK, what letter is it?

Children: W.

Laura: It’s a W. This is an amazing little discovery we’ve got going on here this morning. OK, hands down please.

During this part of the lesson one of the children saw a very different feature of the shapes they were drawing than that which Laura had anticipated. She appeared to use a stalling, rather defensive strategy, not completely dismissing the idea, but later embracing it in a much more positive way. This type of totally unexpected response can cause real difficulties to the teacher if they cannot follow the reasoning or ideas that the child is suggesting. The teacher may be anxious about losing control of the flow of ideas, even that their intended curricular objectives may be completely subverted. Willingness to explore such unexpected contributions is supported by secure content knowledge (both substantive *and* indeed, syntactic).

A final example of a child’s response to an activity having a major impact on the course of part of the lesson comes from Chantal’s first videotaped lesson with a Year 1 class. At one point during the oral and mental starter to the lesson, the class was split into two groups and proceeded to count in ones, with the two groups alternating. A child noticed that the class would be counting in odd and evens, as they had done the previous day. Although Chantal had not raised this with the children,

she developed it further with them. She discussed which numbers are odd and which even, concentrating on the final digits. She followed this through by looking at the 1 to 100 number square, and then wrote the odd and even end-digits on the board. Chantal gave some two-digit examples and children had to say whether they were odd or even. Having told the children to begin counting at 1 (odd), a child asked if they can start at zero. Chantal then suggested they start at zero, which she told them is even. When they had finished counting, Chantal asked each group what sort of numbers their counting ended in.

The whole of this part of the lesson was transformed because of the response of one child to the counting activity. Chantal was able to capitalise on this, and to develop the activity to build on the idea of odd and even within the counting activity.

These examples give a clear picture of the essential relationship between teacher, child and subject. The nature of that relationship can change as a consequence of the contribution of the child to the lesson. We have argued that trainees' response to children's articulated ideas is enabled (or otherwise) by their content knowledge for teaching.

CHOICE OF EXAMPLES

It is helpful to distinguish two rather different uses of examples in teaching. The first is essentially inductive - providing (or motivating students to provide) examples *of* something. The 'something' is *general* in character (e.g. the notion of line symmetry, or the fact that the sum of two odd integers is even); the examples are *particular* instances of the generality. The use of examples to embody abstract concepts and to generalise procedures is commonplace pedagogical practice. Thus, we teach a (general) procedure by a (particular) performance of that procedure. [This aspect of teaching has unhelpfully been called 'modelling' in recent numeracy jargon. There are sufficiently many different meanings of 'model' to insist that we don't need another; 'demonstrate' would do perfectly well in this instance]. For example, if we set out to teach subtraction by decomposition, we might perform, say, $62-38$ in column format. It is important to note that the 6, the 2, the 3 and the 8 were *all* chosen with care in the previous sentence. The provision of such examples is not an arbitrary matter. That is not to say that there is not usually some latitude in the choice of (good) examples. The 8 could have been a 9; on the other hand, it could not have been a 2. It *could* have been a 4, say, but arguably the choice of 4 is pedagogically less good than 8 or 9. Why? Because that would entail the learner in subtracting 4 from 12, a task that would require some pupils to engage in finger-counting, and distract them from the procedure they are meant to be learning.

In the case of concepts, the role of examples is to provoke or facilitate abstraction: once a set of examples has been unified by the formation *of* a concept, subsequent examples can be assimilated *by* the concept (Skemp, 1979). Once a concept has been formed and named by an individual, s/he is able to do something very remarkable - to entertain examples of it outside the realm of personal experience (Rowland, 1999, p.

27); Skemp (*op. cit.*) calls this psychological phenomenon ‘reflective extrapolation’. A teacher’s choice of examples for the purpose of abstraction will reflect his/her awareness of the nature of the concept and the category of things that it comprehends. One of us (TR) is still embarrassed to recall a lesson that he once taught on line symmetry. More than one student did not recognise line symmetry in the letter **E**, since every example he had given had had a vertical axis of symmetry!

A third, far less common, pedagogical deployment of examples in the broadly inductive sense is the presentation of *arguments* through the medium of particular cases, for the purpose of explanation. The construction of such ‘generic examples’ is a matter of considerable pedagogic skill, either conscious or unconscious (Rowland, 2001). The proof of the aforementioned “odd plus odd” property by means of a diagram is a paradigm of this proof strategy.

The second use of examples in teaching, more often called ‘exercises’, is not inductive, but illustrative and *practice*-oriented. We note here that exercises are examples, selected from a class of possible such examples. In the case of two-digit subtraction, 20 exercises might be chosen from the class of some 4000 possible examples. Why choose one subset in preference to another? Characteristically, having learned a procedure (e.g. to add 9, to find equivalent fractions, to find the ‘difference’ of two integers), the student rehearses it on several such ‘exercise’ examples. This is first in order to assist retention of the procedure by repetition, then later to develop fluency with it. Such exercises are also, invariably, an instrument for assessment, from the teacher’s perspective. We recognise that such ‘mere’ practice might also lead to different kinds of awareness and comprehension (just as repeated rehearsal of the notes of a violin concerto might awaken new constructions of the ‘meaning’ of the piece). Again, the selection of such examples by teachers is neither trivial nor arbitrary. The argument for examples to be ‘graded’ is generally well understood, so that students experience success with routine examples before trying more challenging ones. Exercise examples ‘for practice’ will also ideally expose the learner to the range of types of problem that s/he might encounter from time to time. For instance, practice examples on subtraction by decomposition (if we were to insist on teaching it) ought to include some possibilities for zeros in the minuend e.g. 205-87. Bierhoff (1996) has commented that English primary textbooks are poor examples of pedagogy in their provision of examples, compared with those in Germany and Switzerland, whose authors demonstrate far greater didactic awareness.

In both senses of the word, we suggest that the examples provided by a teacher ought, ideally, to be the outcome of a reflective process of *choice*, a *deliberate* and informed selection from the available options, some ‘better’ than others in the sense illustrated above. While we do not pretend to be able to infer such a process of choice, or the lack of it, from the evidence of the videotapes, we can comment on the examples actually chosen by trainees, and how they compare with available alternatives.

The Trainees' Choice of Examples

Whilst we looked for instances of both good and poor choices of examples, the latter seemed to be more prevalent. To redress the balance of this section, therefore, we begin by citing a somewhat isolated (though not unique) instance that seems to us to draw on some key aspects of mathematics content knowledge.

Naomi's lesson was with a Year 1 class. The main activity was about the meaning of the 'difference' of two numbers within 20. Naomi had scored perfect 4s on the audit, yet there is little evidence of overt SMK in the lesson. The lesson would be rich material for a case study, but this is not our purpose here. Suffice it to note the following episode from the mental and oral starter, where the children practised bonds to 10. They sat in a circle, and Naomi chose particular individuals to answer questions such as "If we have nine, how many more to make 10?" The sequence of starting numbers was 8, 5, 7, 4, 10, 8, 2, 1, 7, 3. This seems to us to be a good sequence, for the following reason. The first and third numbers are themselves close to 10, and require little or no counting. 5 evokes a well-known double - doubling being an explicit NNS strategy. The choice of 4 seemed (from the videotape) to be tailored to one of the more fluent children. The degenerate case $10+0$ merits the children's attention. One wonders, at first, why Naomi then returned to 8. The child (Bill) rapidly answers '2'. The answer to our question becomes apparent when Naomi asks the next child, Owen, what he must add to 2 to make 10. Owen counts from 2 on his fingers, and declares '8'. This somewhat drawn-out process proceeds as follows.

Naomi: Owen. Two.

(12 second pause while Owen counts his fingers)

Naomi: I've got two. How many more to make ten?

Owen: (six seconds later) Eight.

Naomi: Good boy. (Addressing the next child). One.

Child: (after 7 seconds of fluent finger counting) Nine.

Naomi: Good. Owen, what did you notice ... what did you say makes ten?

Owen: Um ... four ...

Naomi: You said two add eight. Bill, what did you say? I gave you eight.

Bill: (inaudible)

Naomi: Eight and two, two and eight, it's the same thing.

Naomi's reason for asking about $1+x$ immediately after $2+x$ is not apparent. It could be justified in terms of one less, one more, but Naomi does not draw out this relationship. Instead, Naomi returns to Owen, to ask whether he had noticed the last-but-one question and Bill's answer, adding "eight add two, two add eight, it's the same thing". Admittedly, the significance of Owen's example is lost on him, or has escaped his memory. Nevertheless, there seems to be some conscious design in

Naomi's sequence. Her choice of examples (a) was at first 'graded' (b) included later an unusual/degenerate case, and (c) finally highlighted a key structural property of addition i.e. commutativity. She draws attention to this relationship yet again in her final choice of 7, then 3, and in her comments on this pair of examples.

As with 'responding to children's ideas', certain key categories of the trainees' choice of examples is beginning to emerge. One is the choice of examples that *obscure the role of the variables* within it. One such case concerns Michael, in a lesson with a Year 4 class. The main activity was about telling the time with analogue and digital clocks. One group was having difficulty with analogue quarter past, half past and quarter to. Michael intervened with this group, showing them first an analogue clock set at six o'clock. He then showed them a quarter past six and half past six. When asked to show half past seven on their clocks, one child put both hands on the 7. The child's inference from Michael's demonstration example (half past six) is clear. Of the twelve possible examples available to exemplify half-past, half past six is arguably the most unhelpful.

Another instance took place in a Year 6 lesson which began with work on co-ordinates. Kirsty began by asking the children for a definition of co-ordinates. (The place of definitions as opposed to examples is a topic in its own right, but not for consideration here). One child volunteered that "the horizontal line is first and then the vertical line." Kirsty then asked children to identify the co-ordinates of points as she marked them on a grid. She reminds them that "the x -axis goes first". Her first example is the point (1, 1), which is clearly ineffective in assessing the children's grasp of the significance of the order.

Several other similar examples were readily identified in the videotaped lessons, where the role of a particular variable in a calculation is obscured by the presence of another variable with the *same* value. Chloe is teaching a Year 1/2 class a strategy for adding and subtracting 9, 11, 19 and 21 i.e. by a suitable adjustment of the tens digit and then by adding or subtracting 1 from the units. She asks one child to demonstrate on a number (1 to 100) square by adding 9 to ... 9. To criticise her choice of starting number (9) may seem somewhat churlish. But it was the *first* example offered in the lesson, and she had some 90 starting numbers to choose from (some of which would be unsuitable for a different reason: we return to this episode in a moment). Before moving on from this category of obscuring the role of variables, we mention Colin (Reception class), who selected 10-5 as the first example of subtracting numbers from 10, and also Naomi (mentioned earlier) whose first example of 'difference' was the difference between 4 and 2. Each of these examples inadvertently invites children to construe that the 'answer' is to be found within the original question.

The more general issue that some examples are pedagogically preferable to others is again illustrated within Chloe's lesson (above) when she demonstrates a strategy for subtracting 19 (i.e.. subtracting 20 and adding 1) on a number square. The usual visual representation would be 'up two, right one', like a knight's move. This is good pedagogy, akin to the identification of diagonal lines in the multiple of 11 and 9

which relate to the place value system of numeration. But Chloe chooses 70 for the starting number in her first example, on the extreme right boundary of the 1 to 100 square. After moving up two squares to 50, there is no 'right one' square: it is then necessary to move down and to the extreme left of the next row.

A second category of poor choices of examples arises from the selection of calculations to illustrate a particular procedure, when *another procedure would be more sensible* to perform those particular calculations. A minor instance occurred in Naomi's lesson on 'difference', where she asked (on a worksheet) for the difference between 11 and 10, expecting them to 'count on' from the lesser of the two numbers. This is akin to giving e.g. 302-299 in a set of exercises on subtraction by decomposition. A more worrisome case concerned Laura's choice of demonstration examples in a her first videotaped lesson, on column multiplication (the standard 2-digit by 1-digit algorithm) with a Year 5 class. Her first example (37×9) is not a bad one (though not the best either), but she then goes on to work through 49×4 , 49×8 and 19×4 . Now, the NNS emphasises the importance of *mental* methods, where possible, and also the importance of choosing the most suitable strategy for any particular calculation. 49×4 , 49×8 and 19×4 can all be more efficiently performed by rounding up, multiplication and compensation e.g. $49 \times 4 = (50 \times 4) - 4$. For that matter, 49×8 is readily found by doubling the answer to 49×4 . As we mentioned earlier, the NNS makes much of doubling strategies, and 19×4 could be a double-double. In any case, to carry out these calculations by column multiplication flies in the face of any messages about selecting 'sensible' strategies.

Finally, we note that the videotapes offer copious instances of *examples being randomly generated*, typically by dice. This may have a limited but useful place in the generation of practice exercises, but it is pedagogically perilous in the teaching of procedures or concepts, when, as we have argued, it is simply not the case that any example is as good as any other. The example of subtracting 5 from 10 (in Colin's lesson mentioned earlier) was generated in this way, using specially modified dice. Colin went on to generate further expository examples - 5, 3, 8, in that order - with the dice. This contrasts with Naomi's skilful control of the examples in an episode with a closely-related learning objective (bonds to 10) described above. There seems to be some confusion in the minds of many trainees between the legitimate random choice of examples to enhance conviction about the truth of some principle or the efficacy of some established procedure on the one hand, and the choice of examples to inculcate awareness of a procedure or concept in the first place on the other. The latter is often better controlled and determined by the teacher, and random selection of examples in this case is effectively an abdication of responsibility.

This evidence from our research has greatly enhanced our own awareness that novice teachers need guidance and help in appreciating the different roles of examples in mathematics teaching. The extent to which trainees choose examples wisely, or otherwise, seems to us to be a significant indicator of their mathematics content knowledge for teaching.

CONCLUSIONS

One aim of this research has been to gain insight into a statistical link between trainees' mathematical knowledge and their teaching competence. At this point in time we are some way short of this goal, because reliability issues in the coding of the observed lessons have not yet been resolved. Nevertheless, we have made significant progress towards a related objective: to identify ways in which trainees' subject knowledge, or the lack of it, is evident in their teaching.

Our grounded approach to the analysis of the lessons has highlighted several normative notions of teaching practice, three of which we have elaborated in this paper. These three were chosen from the 18 categories that arose from our scrutiny of the data. Our choice of these particular three categories was guided by two factors. First, as we have already indicated, the importance of *connecting*, *responding* and *exemplifying* is widely supported in the literature. Indeed, each of these is arguably a *sine qua non* of teaching. But even if this is putting the matter too strongly, these are, without doubt, critical indicators of successful teaching. Secondly, the three codes associated with these categories were noticeably prevalent in the *Analytic Accounts* of the 24 lessons. One benefit of observing so many lessons was the realisation that the significance of these three aspects of mathematics teaching could be observed in most of them.

Clearly, our work has not simply confirmed the importance of these three categories which are already upheld in the literature. More importantly, it refines and illuminates them by reference to the classroom practices of novice teachers. So whilst formerly we might have spoken about the importance of responding to children's ideas, for example, we are now able to give a more differentiated account of when such opportunities might occur, and to give examples. In this regard, our own appreciation of their significance has been substantially enhanced.

Finally, we emphasise our narrow attention to mathematics *content* knowledge, both SMK and PCK, in our scrutiny of the lessons. Whilst our research has not provided a direct mapping between mathematical knowledge and competence at teaching the subject, it does throw considerable light on this link by way of particularising from cases.

Acknowledgements

We acknowledge with thanks the considerable contribution of Fay Turner and Jane Webster to this project.

REFERENCES

- Askew, M., Brown, M., Rhodes, V., Wiliam, D., and Johnson, D., (1997) *Effective Teachers of Numeracy: Report of a Study Carried out for the Teacher Training Agency*. London: King's College, University of London.
- Aubrey, C. (1997) *Mathematics Teaching in the early years: an investigation into teachers' subject knowledge*. London: The Falmer Press.

- Ball, D. L. (1990a) 'Prospective elementary and secondary teachers' understanding of division.' *Journal for Research in Mathematics Education*, 21(2), pp. 132-144.
- Ball, D. L. (1990b) 'The Mathematical understandings that prospective teachers bring to teacher education.' *Elementary School Journal*, 90(4), pp. 449-466.
- Barber, P., Heal, C. and Martyn, S. (2002) 'Primary teachers' mathematics content knowledge : progress in developing systems of support?'. Paper given at the Annual Conference of the British Educational Research Association, University of Exeter.
- Barnes, J (1984) *The Complete Works of Aristotle*, Princeton: Princeton University Press
- Begle, E. G. (1968) 'Curriculum research in mathematics.' In H. J. Klausmeier and G. T. O'Hearn (Eds.), *Research and development toward the improvement of education*. Madison, WI: Dembar Educational Research Services.
- Bierhoff, H. (1996) *Laying the Foundations of Numeracy*. London: National Institute of Economic and Social Research, Discussion Paper no. 90.
- Brissenden, T. (1985) 'Children's Language and Mathematics: Rewriting the lesson script.' *Mathematics in School* 14 (4), pp. 2-6.
- Brown, G. and Wragg, E. C. (1993) *Questioning*. London: Routledge.
- DES (1982) *Mathematics Counts*. London: HMSO.
- Dewey, J. (1904). 'The relation of theory to practice in education.' In National Society for the Study of Education (Ed.), *The Relation of Theory to Practice in the Education of Teachers. Third Yearbook. Part I*. Bloomington, IL: Public School Publishing Co.
- DfEE (1999) *The National Numeracy Strategy: framework for teaching mathematics from Reception to Year 6*. Sudbury: DfEE Publications.
- Dickson, L, Brown, M and Gibson, O. (1984) *Children Learning Mathematics*, Eastbourne, Holt Education.
- Eisenberg, T. (1977) 'Begle revisited: teacher knowledge and student achievement in algebra'. *Journal for Research in Mathematics Education* 8, pp. 216-222.
- Gadamer, H. G. (1990) *Truth and Method*. New York: Continuum.
- Glaser, B. G. and Strauss, A. L. (1967) *The Discovery of Grounded Theory: Strategies for Qualitative Research*. New York: Aldine de Gruyter.
- Goulding, M. (2002) P 'Primary teacher trainees' self assessment of their mathematical subject knowledge.' Paper given at the Annual Conference of the British Educational Research Association, University of Exeter.

- Goulding, M. and Suggate, J. (2001) 'Opening a can of worms: investigating primary teachers' subject knowledge in mathematics.' *Mathematics Education Review* 13, pp. 41-54.
- Goulding, M., Rowland, T. and Barber, P. (2002, forthcoming) 'Does it matter? Primary Teacher trainees' subject knowledge in mathematics'. Special Issue of the *British Educational Research Journal* on Subject Matter Knowledge and Application.
- Haylock, D. W (1982) 'Understanding in Mathematics: making connections'. *Mathematics Teaching* MT 92
- Jones, K. and Mooney, C. (2002) 'Trainee primary teachers' knowledge of geometry for teaching'. *Proceedings of the British Society for Research into Learning Mathematics* 22(2), pp. 95-100.
- McNamara, D. (1991) 'Subject knowledge and its application: problems and possibilities for teacher educators.' *Journal of Education for Teaching*, 17(2), pp. 113-128.
- Morris H. J. (2001) 'Issues raised by testing trainee primary teachers' mathematical knowledge'. *Mathematics Teacher Education and Development* 3, pp. 37-48.
- Passmore, J. (1980) *The Philosophy of Teaching*. London: Duckworth.
- Plato (1956 edn) *Protagoras and Meno*. Harmondsworth: Penguin.
- Rowland, T., Martyn, S., Barber, P. and Heal, C. (2000) 'Primary teacher trainees' mathematics subject knowledge and classroom performance' In T. Rowland and C. Morgan (Eds) *Research in Mathematics Education Volume 2: Papers of the British Society for Research into Learning Mathematics*. London: British Society for Research into Learning Mathematics, pp. 3-18.
- Rowland, T., Martyn, S., Barber, P. and Heal, C. (2001) 'Investigating the mathematics subject matter knowledge of pre-service elementary school teachers'. In M. van den Heuvel-Panhuizen (Ed.) *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, Volume 4 pp. 121-128. Utrecht, The Netherlands, Freudenthal Institute, Utrecht University
- Rowland, T., Martyn, S., Barber, P. and Heal, C. (2002) 'The mathematics subject knowledge of pre-service elementary school teachers'. In D. Edge and B. H. Yeap (Eds.) *Mathematics Education in a Knowledge-Based Era: Proceedings of the Second East Asian Regional Conference on Mathematics Education and Ninth Southeast Asian Conference on Mathematics Education* Volume 2, pp. 274-280. Singapore: Association of Mathematics Educators/Nanyang Technological University.
- Sanders S. E. and Morris H. J. (2000) 'Exposing student teachers' content knowledge: empowerment or debilitation?' *Educational Studies* 26(4), pp. 397-408.

- Shulman, L. (1986) 'Those who understand, knowledge growth in teaching'.
Educational Researcher 15(2), pp 4-14.
- Suggate, J, Davis, A. and Goulding, M. (1998) *Mathematical Knowledge for Primary Teachers* David Fulton.
- Wilson, J. (1975) *Education Theory and the Preparation of Teachers*. Windsor:
NFER.
- Wragg, E. C. and Brown, G. (1993) *Explaining*. London: Routledge.