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ELEMENTARY TEACHERS' MATHEMATICS SUBJECT KNOWLEDGE: THE KNOWLEDGE QUARTET AND THE CASE OF NAOMI

ABSTRACT. This paper draws on videotapes of mathematics lessons prepared and conducted by pre-service elementary teachers towards the end of their initial training at one university. The aim was to locate ways in which they drew on their knowledge of mathematics and mathematics pedagogy in their teaching. A grounded approach to data analysis led to the identification of a 'knowledge quartet', with four broad dimensions, or 'units', through which mathematics-related knowledge of these beginning teachers could be observed in practice. We term the four units: foundation, transformation, connection and contingency. This paper describes how each of these units is characterised and analyses one of the videotaped lessons, showing how each dimension of the quartet can be identified in the lesson. We claim that the quartet can be used as a framework for lesson observation and for mathematics teaching development.

KEY WORDS: elementary teaching, mathematics, teacher education, teacher knowledge, videotape

INTRODUCTION

The seven categories of teacher knowledge identified in the seminal work of Lee Shulman include three with an explicit focus on 'content' knowledge: subject matter knowledge, pedagogical content knowledge and curricular knowledge. Subject matter knowledge (SMK) is knowledge of the content of the discipline *per se* (Shulman, 1986, p. 9), consisting both of substantive knowledge (the key facts, concepts, principles and explanatory frameworks in a discipline) and syntactic knowledge (the nature of enquiry in the field, and how new knowledge is introduced and accepted in that community).

Pedagogical content knowledge (PCK) is particularly difficult to define and characterise, conceptualising both the link and the distinction between knowing something for oneself and being able to enable others to know it. PCK consists of "the ways of representing the subject which makes it comprehensible to others...[it] also includes an understanding of what makes the learning of specific topics easy or difficult ..." (Shulman, 1986, p. 9). Curricular knowledge encompasses the scope and sequence of teaching programmes and the materials used in them.

An uninformed perspective on SMK in relation to mathematics teaching might be characterised by the statement that secondary teachers already have it and elementary teachers need very little of it. There is evidence from the UK and beyond to refute both parts of that statement (e.g. Alexander, Rose & Woodhead, 1992; Ball, 1990a, b; Ma, 1999; Ofsted, 1994). Ma, in particular, presents compelling evidence that the adequacy of elementary teachers' substantive and syntactic knowledge of mathematics, for their own professional purposes, cannot by any means be taken for granted. Recent government initiatives to enhance the mathematics SMK and PCK of prospective and serving elementary teachers have been taken in a number of countries e.g. England (DfEE, 1998; TTA, 2002), Israel (Tsamir & Tirosh, 2003). The research reported here took place against the background of a UK government circular (DfEE, 1998) which specifies a curriculum for Initial Teacher Training (ITT) in England. This circular includes a specification of what its authors deem to be the "knowledge and understanding of mathematics that trainees[1] need in order to underpin effective teaching of mathematics at primary[2] level". (p. 48)

This paper is located in a collaborative project involving researchers at three UK universities, under the acronym *SKIMA* (subject knowledge in mathematics). The conceptualisation of subject knowledge which informs the project and its relation to teaching has been detailed elsewhere (Goulding, Rowland & Barber, 2002).

The focus of the research reported in this paper is on ways that elementary trainees' mathematics content knowledge, both SMK and PCK, can be seen to contribute to their teaching during the 'practical' element of their training – the school-based placements. The research had multiple objectives, the first of which was to develop an empirically-based conceptual framework for productive discussion of mathematics content knowledge between teacher educators, trainees and teacher-mentors, in the context of school-based placements. Such a framework would need to be manageable, and not overburdened with structural complexity. It would need to capture a number of important ideas and factors about content knowledge within a small number of conceptual categories, with an equally small set of easily-remembered labels for those categories. The path that we followed to achieve this objective is the subject of this paper.

We wish to clarify at the outset that our approach, and the application of our research to teacher education inherent in our stated objective, is about raising awareness; it is *not* about being judgemental. Whilst we see certain kinds of knowledge to be *desirable* for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, *ought* to know. We return to this matter toward the end of this paper. Our interest is in what a teacher *does* know and believe, and how opportunities to enhance knowledge can be identified. We believe that the framework that arose from this research – we call it the 'knowledge quartet' – provides a means of reflecting on teaching and teacher knowledge, with a view to developing both.

METHOD

In the UK, the vast majority of trainees follow a one-year, full-time course leading to a Postgraduate Certificate in Education (PGCE) in a university education department, about half the year being spent working in a school under the guidance of a school-based mentor. The final phase of this school-based component takes place towards the end of the course, and is assessed on a simple Pass/Fail basis, a Pass being one of several requirements for successful teacher certification on conclusion of the course. All primary trainees are trained to be generalist teachers of the whole primary curriculum. Later in their careers, most take on responsibility for leadership in one curriculum area (such as mathematics) in their school, but, almost without exception, they remain generalists, teaching the whole curriculum to one class.

This study took place in the context of such a one-year PGCE course, in which each of the 149 trainees followed a route focusing either on the 'lower primary' years (LP, ages 3–8) or the 'upper primary' (UP, ages 7–11). Two mathematics lessons taught by six LP and six UP trainees were observed and videotaped i.e., 24 lessons in total. Trainees were asked to provide a copy of their planning for the observed lesson. As soon as possible after the lesson (usually the same day) the observer/researcher wrote a *Descriptive Synopsis* of the lesson. This was a brief (400–500 words) account of what happened in the lesson, so that a reader might immediately be able to contextualise subsequent discussion of any events within it. These descriptive synopses were typically written from memory and field notes, with occasional reference to the videotape if necessary.

From that point, we took a grounded approach to the data for the purpose of generating theory (Glaser and Strauss, 1967). In particular, we identified in the videotaped lessons aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics content knowledge or their mathematical pedagogical knowledge. These were grounded in particular moments or episodes in the tapes. Our analyses took place after the assessment of the school-based placement had taken place, and played no part in the assessment process. Our task was to look for issues relating to the trainees' mathematics SMK and PCK, and not to make summative assessments of teaching competence.

This inductive process generated a set of 18 codes. Next, we revisited each lesson in turn and, after further intensive study of the tapes, elaborated each *Descriptive Synopsis* into an *Analytical Account* of the lesson. In these accounts, significant moments and episodes were identified and coded, with appropriate justification and analysis concerning the role of the trainee's content knowledge in the identified passages, with links to relevant literature.

Codes, Categories and Superordinate Classes

Our catalogue of 18 codes presented us with a relatively fine-grained dissection of the elementary mathematics teaching that we observed, with specific reference to the contribution of mathematics content knowledge, both SMK and PCK. This was useful to the extent that we had a set of concepts and an associated vocabulary sufficient to identify and describe various ways in which mathematics content knowledge influenced the choices and actions of these novice teachers in their classrooms.

At the same time, the identification of these fine categories was a stepping stone to our intention to offer them to colleagues for their use, as a framework for reviewing mathematics teaching with trainees following observation. We did not want an 18-point tick-list (like an annual car safety check), but a readily-understood scheme which would serve to frame an in-depth *discussion* between teacher and observer. The key to our solution was the recognition of an association between elements of subsets of the 18 codes. This enabled us to group the 18 categories into four broad, superordinate categories, or 'units'. These four units are the dimensions or 'members' of what we call the 'knowledge quartet'. We have named these units as follows:

– foundation;
– transformation;
– connection;
– contingency.

Each unit is composed of a small number of subcategories that we judged, after extended discussions, to be of the same or a similar nature. For example, the third unit, *connection*, is a synthesis of four codes, namely: *making connections; decisions about sequencing; anticipation of complexity*, and*recognition of conceptual appropriateness*. At the same time, we believe that our names for these original ingredients – the codes for the subcategories of each unit – are less important to other users of the 'quartet' than a broad sense of the general character and distinguishing features of each of the units. We shall attempt to enable readers to 'get a sense' of the four units in a moment.

Our research suggests that the quartet is comprehensive as a tool for thinking about the ways that subject knowledge comes into play in the classroom. However, it will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four units; for example, a *contingent* response to a pupil's suggestion might helpfully *connect* with ideas considered earlier. Furthermore, it will become clear that the application of subject knowledge in the classroom always rests on foundational knowledge; for this reason we devote most space later in this paper to the consideration of foundational knowledge.

Exemplification and Conceptualisation: The Case Study

The process by which we arrived at the four dimensions of the knowledge quartet was grounded and inductive, by constant comparison across 24 lessons. A full account of this process, and an argument for the validity and generality of the framework – the quartet – is to be made in papers written concurrently with this one. Our purpose in this paper is more illustrative than inductive. In the next section, we shall give a succinct account of how we conceptualise the character of each dimension of the quartet. Then we shall show how and where it can be applied by detailed reference to one of the 24 videotaped lessons. The trainee in question, Naomi, was teaching a class of 5- and 6-year olds about subtraction. By focusing on just one lesson we hope to maximise the possibility of the reader's achieving some familiarity with the context for the analysis, including Naomi and the children in her class, as well as the structure and the flow of her lesson.

THE KNOWLEDGE QUARTET

The brief conceptualisation of the knowledge quartet which now follows draws on the extensive range of data referred to above. Some aspects of the characterisation below will emerge from our consideration of the lesson that we have singled out for attention later in this paper.

Foundation

The first member of the quartet is rooted in the foundation of the trainees' theoretical background and beliefs. It concerns trainees' knowledge, understanding and ready recourse to their learning *in the academy*, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units in the sense that it is about knowledge possessed, irrespective of whether it is being put to purposeful use. This distinction relates directly to Aristotle's account of 'potential' and 'actual' knowledge. "A man is a scientist ... even when he is not engaged in theorising, provided that he is capable of theorising. In the case when he is, we say that he is a scientist in actuality." (Lawson-Tancred, 1998, p. 267). Both empirical and theoretical considerations have led us to the view that the other three units flow from a foundational underpinning.

A key feature of this category is its *propositional* form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics *per se*; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt.

As regards knowledge of mathematics, we include aspects of 'knowing why' with regard to the topic in hand. Ma (1999), and Ball (1990a) before her, have exposed the inadequacy of procedural, instrumental understanding even of elementary topics for teaching. A rather less fundamentally mathematical aspect of knowledge in this category is the careful and deliberate (we hold back from the word 'correct') use of mathematical vocabulary.

Regarding knowledge of mathematics education, we identify here what might be regarded as aspects of research of a fundamental and far-reaching kind. For example, the UK has seen a recent, seismic shift in the foundation of Early Years mathematics from sets, one–one correspondence and cardinality to counting and ordinality, along with shifts of emphasis from partition and place-value to holistic, sequential perspectives, and from written algorithms to flexible mental methods (e.g. Thompson, 1997). Knowledge of key didactical principles has the potential to inform lesson planning, whilst leaving open to deliberation the details of exposition and task design.

The *beliefs* component of this category is different in kind. We see three aspects of this component. The first has to do with beliefs about the nature of mathematics itself and different philosophical positions regarding the nature of mathematical knowledge (e.g. Hersh, 1997). Secondly, we identify beliefs about the *purposes* of mathematics education, and why particular mathematics topics should be studied in school (e.g. Bramall & White, 2000). Thirdly, teachers hold different beliefs about the conditions under which pupils will best learn mathematics. In fact, there is compelling evidence to suggest that experiences as a learner of mathematics, beliefs about the nature of mathematics and instructional practices as a teacher of mathematics are interconnected (Lampert, 1988; Lerman, 1990; Sanders, 1994; Thompson, 1984, 1992).

In summary, this category that we call 'foundation' coincides to a significant degree with what Shulman (1987) calls 'comprehension', being the first stage of his six-point cycle of pedagogical reasoning.

Transformation

The remaining three categories, unlike the first, refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching. They focus on knowledge-in-action as *demonstrated* both in planning to teach and in the act of teaching itself. At the heart of the second member of the quartet, and acknowledged in the particular way that we name it, is Shulman's observation that the knowledge base for teaching is distinguished by " ... the capacity of a teacher to *transform* the content knowledge he or she possesses into forms that are pedagogically powerful" (1987, p. 15, emphasis added). This characterisation has been echoed in the writing of Ball (1988), for example, who distinguishes between knowing some mathematics 'for yourself' and knowing in order to be able to help someone else learn it. As Shulman indicates, the presentation of ideas to learners entails their re-presentation (our hyphen) in the form of analogies, illustrations, examples, explanations and demonstrations (Shulman, 1986, p. 9).

Our second category, unlike the first, picks out behaviour that is directed towards a pupil (or a group of pupils), and which follows from deliberation and judgement informed by foundation knowledge. This category, as well as the first, is informed by particular kinds of literature. We recognise that trainees turn to different kinds of guidance and inspiration in the pressured environment of the school-based placement. This might be found in the teachers' handbooks of textbook series, in the articles and 'resources' pages of professional journals, in long-established primary mathematics teaching manuals such as Williams and Shuard (1994) or in the newer generation of handbooks such as Askew (1998). Increasingly, in the UK, teachers look to the internet for bright ideas and even for ready-made lesson plans. The trainees' choice and *use of examples* has emerged as a rich vein for reflection and critique. This includes the use of examples to assist concept formation, to demonstrate procedures, and the selection of exercise examples for student activity.

Connection

The next category binds together certain choices and decisions that are made for the more or less discrete parts of mathematical content – the learning, perhaps, of a concept or procedure. It concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons. Mathematics is notable for its coherence as a body of knowledge and as a field of enquiry, and the cement that holds it together is reason. Russell and others even attempted to demonstrate, albeit unsuccessfully, that mathematics can be reduced to pure logic (Ernest, 1998).

The pursuit of coherence and mathematical connections in mathematics pedagogy has been stimulated recently by the work of Askew, Brown, Rhodes, Wiliam, and Johnson (1997); of six case study teachers found to be highly effective, all but one gave evidence of a 'connectionist' orientation. The association between teaching effectiveness and a set of articulated beliefs of this kind lends a different perspective to the work of Ball (1990b) who also strenuously argued for the importance of connected knowledge for teaching. In a discussion of 'profound understanding of fundamental mathematics', Ma cites Duckworth's observation that intellectual 'depth' and 'breadth' "is a matter of making connections" (Ma, 1999, p. 121).

In addition to the integrity of mathematical content in the mind of the teacher and his/her management of mathematical discourse in the classroom, our conception of coherence includes the *sequencing* of topics of instruction within and between lessons, including the ordering of tasks and exercises. To a significant extent, these reflect deliberations and choices entailing not only knowledge of structural connections within mathematics itself, but also awareness of the relative cognitive demands of different topics and tasks.

Contingency

The fourth member of the quartet is distinguished both from the possession of a theoretical background, on the one hand, and from the deliberation and judgement involved in making learning meaningful and connected for pupils, on the other. Our final category concerns classroom events that are almost impossible to plan for. In commonplace language it is the ability to 'think on one's feet': it is about *contingent action*. The two constituent components of this category that arise from the data are the readiness to *respond to children's ideas* and a consequent preparedness, when appropriate, to *deviate from an agenda* set out when the lesson was prepared. Shulman (1987) proposes that most teaching begins from some form of 'text' – a textbook, a syllabus, ultimately a sequence of planned, intended actions to be carried out by the teacher and/or the students within a lesson or unit of some kind. Whilst the stimulus – the teacher's intended actions – can be planned, the students' responses cannot.

A constructivist view of learning provides a valuable perspective on children's contributions within lessons. When a child articulates an idea, this points to the nature of *their* knowledge construction, which may or may not be quite what the teacher intended or anticipated. The child's indications of their meaning-making, spoken or written, may be elicited by responses to exercises, by a teacher's questions, or may be volunteered unsolicited. To put aside such indications, or simply to ignore them or dismiss them as 'wrong', can be construed as a lack of interest in what it is that that child (and possibly others) have come to know as a consequence, in part, of the teacher's teaching.

Brown and Wragg (1993) group listening and responding together in a taxonomy of 'tactics' of effective questioning. They observe that "our capacity to listen diminishes with anxiety" (p. 20). Uncertainty about the sufficiency of one's subject matter knowledge may well induce such anxiety, although this is just one of many possible causes.

Brown and Wragg add that 'responding' moves are the lynch pins of a lesson, important in the sequencing and structuring of a lesson, and observe that such interventions are some of the most difficult tactics for newly qualified teachers to master. The quality of such responses is undoubtedly determined, at least in part, by the knowledge resource available to the teacher. For example, Bishop (2001, pp. 95–96) recounts a nice anecdote about a class of 9- and 10-year-olds who were asked to give a fraction between 1/2 and 3/4. One girl answered 2/3, "because 2 is between the 1 and the 3, and on the bottom the 3 lies between the 2 and the 4". Bishop asks his readers how they might respond to the pupil. It is relevant here to suggest that such a response might be conditioned by whether they were aware of Farey sequences and mediants, or what heuristics were available to them to explore the generalisation inherent in the pupil's justification.

Our conceptualisation of the four members of the knowledge quartet is briefly summarised in Table I.

NAOMI

We come now to our case study. Naomi had chosen the Lower Primary specialism. At school, she had achieved an A* GCSE[3] mathematics grade – the highest possible, achieved by two percent of candidates nationally. She had chosen Mathematics as one of her three Advanced Level GCE[4] subjects, achieving a B grade. Her undergraduate degree study had been in Philosophy: she had been exposed to no academic study of mathematics in the four years immediately preceding her PGCE.

We offer Naomi's lesson as a 'case' in the following sense: it is typical of the way that the quartet can be used to identify, for discussion, matters that arise from the lesson observation, and to structure reflection on the lesson.

Naomi's Lesson

This was Naomi's first videotaped lesson with her Year 1[5] class. The number of pupils is not noted in the lesson synopsis, but one can see around 20 in the class. Naomi's classroom is bright and spacious, with a large open, carpeted area. The learning objectives stated in Naomi's lesson plan are as follows: "To understand subtraction as 'difference'. For more able pupils, to find small differences by counting on. Vocabulary – *difference, how many more than, take away*." Naomi notes in

TABLE I
The Knowledge Quartet

	The Knowledge Quartet
Foundation	Propositional knowledge and beliefs concerning: • the meanings and descriptions of relevant mathematical concepts, and of relationships between them; • the multiple factors which research has revealed to be significant in the teaching and learning of mathematics; • the ontological status of mathematics and the purposes of teaching it. Contributory codes: awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures
Transformation	Knowledge-in-action as revealed in deliberation and choice in planning and teaching. The teacher's own meanings and descriptions are transformed and presented in ways designed to enable students to learn it. These ways include the use of powerful analogies, illustrations, explanations and demonstrations The choice of examples made by the teacher is especially visible: • for the optimal acquisition of mathematical concepts, procedures or essential vocabulary; • for confronting and resolving common misconceptions; • for the justification (by generic example) or refutation (by counter-example) of mathematical conjectures. Contributory codes: choice of representation; teacher demonstration; choice of examples
Connection	Knowledge-in-action as revealed in deliberation and choice in planning and teaching. Within a single lesson, or across a series of lessons, the teacher *unifies* the subject matter and draws out *coherence* with respect to: • connections between different meanings and descriptions of particular concepts *or* between alternative ways of representing concepts and carrying out procedures; • the relative complexity and cognitive demands of mathematical concepts and procedures, by attention to sequencing of the content. Contributory codes: making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness

TABLE I
(Continued)

	The Knowledge Quartet
Contingency	Knowledge-in-*inter*action as revealed by the ability of the teacher to 'think on her feet' and respond appropriately to the contributions made by her students during a teaching episode. On occasion this can be seen in the teacher's willingness to deviate from her own agenda when to develop a student's unanticipated contribution: • might be of special benefit to that pupil, *or* • might suggest a particularly fruitful avenue of enquiry for others. Contributory codes: responding to children's ideas; use of opportunities; deviation from agenda

her plan that they have already learnt *how many more than*. An overview of her lesson – essentially our *Descriptive Synopsis* – is as follows.

Naomi settles the class in a rectangular formation around the edge of the carpet in front of her, then the lesson begins with a seven-minute *Mental and Oral Starter*[6] designed to practice number bonds to 10. A 'number bond hat' is passed from child to child until Naomi claps her hands. The child holding, then wearing, the hat is given a number between zero and ten, and required to state how many more are needed to make ten. Later, two numbers are given. The child must add them and say how many more are then needed to make ten.

The *Introduction* to the *Main Activity* lasts nearly 20 minutes. Naomi sets up various 'difference' problems, initially in the context of frogs in two ponds. Her pond has four, her neighbour's has two. Magnetic 'frogs' are lined up on a vertical board, in two neat rows. She asks first how many more frogs she has and then requests the difference between the numbers of frogs. Pairs of children are invited forward to place numbers of frogs (e.g. 5, 4) on the board, and the differences are explained and discussed. Before long, she asks how these differences could be written as a "take away sum". With assistance, a girl, Zara, writes $5-4=1$. Later, Naomi shows how the difference between two numbers can be found by counting on from the smaller.

The children are then assigned their *group tasks*. One group ('whales'), supported by a teaching assistant, is supplied with a worksheet in which various icons (such as cars and apples) are lined up to 'show' the difference, as Naomi had demonstrated with the frogs. Two further groups (Dolphins and Octopuses) have difference word problems (e.g. I have 8 sweets and you have 10 sweets) and are directed to use 'multilink' plastic cubes to solve them, following the 'frogs' pairing procedure. The remaining two groups have a similar problem sheet, but are directed to use the counting-on method to find the differences. To begin with, Naomi moves around the class, working with individuals. It turns out that the children who are using the multilink cubes experience some difficulty applying the intended method. Sixteen minutes into the groupwork phase, Naomi takes the Dolphins and Octopuses back to the carpet and teaches them to use the counting-on method.

Nine minutes later, Naomi calls the class together on the carpet for an eight-minute *Plenary*, in which she uses two large, foam 1–6 dice to generate two numbers, asking the children for the difference each time. Their answers indicate that there is still widespread confusion among the children, in terms of her intended learning outcomes. The one-hour tape runs out, presumably just before the conclusion of the lesson.

THE KNOWLEDGE QUARTET: NAOMI'S LESSON

Earlier, we introduced the four units of the knowledge quartet, and gave a general account of the characteristics of each unit. We now offer our interpretation of some ways in which we have observed or inferred foundation, transformation, connection and contingency in Naomi's first videotaped lesson.

Foundation

Carpenter and Moser (1983) identify four broad types of subtraction problem structure, which they call *change, combine, compare, equalise*. These find their way into the UK practitioner literature as different subtraction 'models' (e.g. Haylock & Cockburn, 1997). Two of these problem types are particularly relevant to Naomi's lesson. First, the change-separate problem, exemplified by Carpenter and Moser as: "Connie had 13 marbles. She gave 5 marbles to Jim. How many marbles does she have left" (p. 16). The UK practitioner language for this is subtraction as 'take away' (DfEE, 1999, p. 5/28).

Secondly, the compare problem type, one version of which is: "Connie has 13 marbles and Jim has 5 marbles. How many more marbles does Connie have than Jim". (Carpenter & Moser, 1983, p. 16). This subtraction problem type has to do with situations in which two sets (Connie's marbles and Jim's) are considered simultaneously; this contrasts with change problems, which involve an action on and transformation of a single set (Connie's marbles). Again, the National Numeracy Strategy *Framework* (DfEE, 1999) reflects the tradition of UK practitioners in referring to the compare structure as 'subtraction as *difference*'.

Carpenter and Moser go on to show that the semantics of problem structure, as discussed above, by no means determines the processes of solution adopted by individual children, although the structure might suggest a paradigm, or canonical, strategy. Some strategies involve actions ('direct modeling') with concrete materials, while others depend on forms of counting. We select and elaborate here the two solution strategies relevant to our analysis of Naomi's lesson. First, *separating from*, the canonical strategy for the change-separate ('take-away') structure described above, involves constructing the minuend set and then removing a number of objects corresponding to the subtrahend. Counting the remaining objects yields the answer. The parallel counting strategy is called *counting down from*. The child counts backwards, beginning with the minuend. The number of iterations in the backward counting sequence is equal to the subtrahend. The last number uttered is the difference. Secondly, Carpenter and Moser's taxonomy of strategies includes *matching*; the child puts out *two* sets of objects with the appropriate cardinalities. The sets are then matched one-to-one; counting (or subitising) the unmatched cubes gives the answer.

The National Numeracy Strategy *Framework* (DfEE, 1999) recommends the introducing of subtraction, first as take-away, in Year R, and then as comparison in Year 1. One consequence of this sequence is the almost universal use of 'take away' as a synonym for subtraction in elementary classroom discourse (Haylock and Cockburn, 1997, p. 38), as we shall see in a moment. Another peculiarly-British complication, as we mentioned earlier, is that the word 'difference' has come to be associated in rather a special way with the comparison *structure* for subtraction. At the same time, the term *difference* is the unique name of the outcome of *any* subtraction operation, on a par with *sum, product* and *quotient* in relation to the other three arithmetic operations.

These theoretical considerations are part of the foundational knowledge relevant to Naomi's lesson. This is not to suggest that Naomi would know or articulate them as we have, though it is clear from her lesson plan that she intends to address 'difference' both conceptually and linguistically. That is to say, she wants the pupils to learn to perceive subtraction in terms of comparison, and to be able to answer appropriately questions about "the difference" between two numbers. Her plan suggests that she is aware of the two models (problem structures) of subtraction discussed above, and the need for children to learn both. In her introduction, she teaches the *matching* strategy, arranging the frogs into two rows in order to facilitate comparison of the two sets.

Naomi: Right. I had four frogs, so I was really pleased about that, but then my neighbour came over. She's got some frogs as well, but she's only got two. How many more frogs have I got? Martin?
Martin: Two.
Naomi: Two. So what's the difference between my pond and her pond in the number of frogs? Jeffrey.
Jeffrey: Um, um, when he had a frog you only had two frogs.
Naomi: What's the difference in number? [...] Martin said I've got two more than him. But we can say that another way. We can say the difference is two frogs. You can take these two and count on three, four, and I've got two extra.

First, Naomi poses the comparison problem in terms of "how many more?", and Martin is able to respond correctly to this formulation. Her next question asks for the difference. The word *difference* has not cued Jeffery as intended; his difficulty with the word is a well-documented case of polysemy (Durkin & Shire, 1991). Naomi has to be more explicit ("we can say that another way") about the connection with the earlier "more than" problem. Note that she introduces the *counting up* subtraction strategy intended for the "more able" in her next utterance (above), although, in this instance, she has already *matched* the first two pairs of frogs, and the remaining two (in 'her pond') can easily be subitised.

Other children also have difficulty making the desired responses to the difference problems. Leo, for example:

Naomi: Martin's got four frogs, one, two, three, four, but how many more does Bill have? How many more? Leo, can you see? How many more does he have?
Leo: Um, six?

Naomi initially responds by drawing attention to the excess after matching:

[Explanation 1] OK, let me explain it this way. Right, looking at me, thank you, Jeffrey, thank you. Martin's got four and I want to know how many more Bill's got. Bill's got four as well, but then he's got the two extra ones. So, what is the difference?

Jared also offers six. At this point, Naomi incorporates again the counting up approach. She explains that:

[Explanation 2] We can do this on our fingers as well. If we start with the small number, four, we can count on the extra two. Right, can everybody show me four fingers. Now, Bill's got six, so you count four, [pause] five, six. We've added on two more, and the total is six. The difference is two. Bill, do you think you know how to write that as a take away sum?

Whether by accident or design, it is the case that Naomi's two explanation types (above) are well-founded in Carpenter and Moser's research.

When they go to their desks to do the word problems, two groups of children ('Octopuses' and 'Dolphins') are provided with manipulatives (the multilink cubes) and directed to use the *matching* strategy that Naomi had demonstrated, using the cubes as surrogate frogs. However, the other two groups (described by Naomi as "more able" in her lesson plan) are guided to solve their word problems by *counting up* from the smaller number.

Naomi's pedagogic *beliefs*, and the way they are challenged in the course of the lesson, are highlighted as we track the progress of the Octopuses and Dolphins. The first worksheet question concerns the difference between 8 sweets and 10 sweets. The two manipulative groups are seen to have difficulty mimicking Naomi's earlier demonstration with the frogs. It is not long before she intervenes. Naomi emphasises putting 8 cubes in a row, then 10. "Then you can *see* what the difference is". She demonstrates again, but none of the children seems to be copying her. Jared can be seen moving the multilink cubes around the table, apparently aimlessly. Another child says "I don't know what to do". Given the competing demands on her time, Naomi moves away to give her attention to the count-up groups. In her absence from the table, one boy sets about constructing a tower with the cubes. Ten minutes later, Naomi returns to the Dolphins group, and attempts once again to clarify the multilink method. She asks "What's the difference between 7 and 12?". Without looking up, the Tower Boy replies, "Don't ask me, I'm too busy building".

This seems to be a turning point for Naomi, who looks exasperated and suddenly declares "Goodness me, let's put these away. I'll show you a different way to do it". She collects up the multilink cubes into a tray, and takes the Dolphins and Octopuses back to the carpet, where she shows them the counting up strategy for the difference between 8 and 10. "You start with the lower number ... you start with the smallest number. Count on – show me your fist – nine, ten". She then works through the first three worksheet questions, doing them for the children, by counting up.

We can safely infer from Naomi's pedagogical *intentions* for the solution of the difference problems that she believed, at the planning stage at least, that the lower attaining children (our language) needed a different computational strategy from the rest of the class i.e. matching – a procedure that *requires* the use of materials (Carpenter & Moser, 1983). Carpenter and Moser found that untaught Grade 1 pupils tend to opt for the matching strategy for comparison problems. Whether that is a recommendation for *teaching* the strategy in Grade 1 is not so clear.

In relation to our stated objective, we might therefore identify for later discussion with Naomi matters such as:

- Did Naomi feel that the multiple meanings of 'difference' were problematic in the lesson?
- Does Naomi perceive a tension in her intention to teach subtraction as difference and her frequent reference to the subtraction operation as 'take away'?
- Is she in sympathy with the NNS guidance to move towards written recording of difference problems with Year 1 pupils? Is the conventional record with '–' and '=' the most helpful?
- What considerations guided her decision to teach these two 'difference' strategies i.e. matching and counting up, and her choice of strategy for each of the groups in the main activity? Why did she abandon matching with the Octopuses and Dolphins? What would she do another time?

Other foundational matters could be identified, of course, but these four would be more than enough for a 20- or 30-minute post-observation discussion.

Transformation

We have already touched, in the previous section, on the analogies, illustrations, examples, explanations and demonstrations that abound in Naomi's lesson. She had made a very deliberate and appropriate choice in her representation of the comparative aspect of whole number subtraction, and taken care over the preparation of the magnetic 'frogs', intended, presumably, to frame her demonstration in a 'real life' context that might interest and motivate them.

Less obvious perhaps, but arguably even more impressive, is her choice of examples in the 'number bond hat' episode in the Mental and Oral Starter. Recall that Naomi chose particular individuals to answer questions such as "If we have nine, how many more to make 10?" Naomi's sequence of starting numbers was 8, 5, 7, 4, 10, 8, 2, 1, 7, 3. This seems to us to be a well-chosen sequence, for the following reason. The first and third numbers are themselves close to 10, and require little or no counting to arrive at the answer. Five evokes a well-known double – doubling being an explicit NNS strategy. The choice of 4 seemed (from the videotape) to be tailored to one of the more fluent children. The degenerate case $10+0$ merits the children's attention. One wonders, at first, why Naomi then returned to 8. The child (Bill) rapidly answers '2'. The answer to our question becomes apparent when Naomi comes to the next child, Owen. The interaction between Naomi and the pupils proceeds as follows.

Naomi:	Owen. Two.
(12 second pause while Owen counts his fingers)	
Naomi:	I've got two. How many more to make ten?
Owen:	(six seconds later) Eight.
Naomi:	Good boy. (Addressing the next child). One.
Child:	(after 7 seconds of fluent finger counting) Nine.
Naomi:	Good. Owen, what did you notice ... what did you say makes ten?
Owen:	Um ... four ...
Naomi:	You said two add eight. Bill, what did you say? I gave you eight.
Bill:	(inaudible)
Naomi:	Eight and two, two and eight, it's the same thing.

Naomi's reason for giving the child after Owen the number 1 (Naomi's third 'turn' above) immediately after Owen's question (about '2') is not apparent. It could be justified in terms of one less, one more, but Naomi does not draw out this relationship. Instead, Naomi returns to Owen, to ask whether he had noticed the last-but-one question and Bill's answer, adding "eight add two, two add eight, it's the same thing". Admittedly, the significance of Owen's example is lost on him, or has escaped his memory. Nevertheless, there seems to be some conscious design in Naomi's sequence. Her choice of examples (a) was at first 'graded' (b) included later an unusual/degenerate case, and (c) finally highlighted a key structural property of addition i.e. commutativity. She draws attention to this relationship yet again in her final choice of 7, then 3, and in her comments on this pair of examples.

It is not our intention, in this paper, to compare or contrast Naomi's lesson with those of the other 11 trainees, but we note that the 24 videotapes offer copious instances of examples being *randomly* generated, typically by dice in a number of situations. Of course, Naomi herself uses dice to generate 'difference' examples in her Plenary. This introduces an example that *obscures the role of variables*, a phenomenon that we observed in several of the videotaped lessons. In this instance, Naomi rolls 3 and 6 on the two dice. Jim appears to have the answer:

Naomi:	Who can do this one for me? What's the difference between three and six? Jim. Jim was sitting quietly, come and tell us.
Jim:	Three.

Only later does it become clear that Jim is not offering an answer to the question, but uttering the first of the two numbers with the intention of adding the second to it. 'Three', is both subtrahend *and* difference in this example. As the tape runs out, Naomi is asking the class for the difference between four and two.

Where transformation is concerned, we might therefore identify for later discussion with Naomi matters such as:

- What factors and considerations led to her strategic choice of examples in the Mental and Oral starter?
- Why did she decide to generate the examples for the plenary randomly? How effective were the strategic and random choices?

Connection

It seems to us that the lesson offers the opportunity for Naomi to make two important connections relative to the considerations that have already been aired with regard to Naomi's foundational knowledge. The first connection is that between the two subtraction structures i.e. change-separate ('take-away') and comparative. The two involve very different procedures when carried out with manipulative materials, and it might not be apparent to pupils that they achieve the same outcome for a given subtraction. There is a dilemma here, and a choice for Naomi, in that the learning objective for this lesson is subtraction-as-difference. Might the children be 'confused' by the inclusion of take-away subtraction in the same lesson? As we have seen, Naomi did, in fact, use the language of 'take-away' throughout the lesson with reference to symbolic recording of the matching procedure. She was, by implication, saying that this procedure (lining up two sets and looking at the excess) was achieving the same result as their previously-learned take-away procedure, since they *recorded* both in the same way i.e. $a–b = c$. Furthermore, as we have already noted, Naomi sometimes reverts to the take-away model in a last-ditch attempt to elicit a correct answer to her difference questions.

The second connection that we have in mind is that between the two strategies for comparison i.e. matching (using manipulatives) and counting up. Naomi implies that there *is* a link between the two when, for example, she says "We can do this on our fingers as well". However, the counting up procedure was only ever intended for the "more able" children, and when, eventually, she teaches it to the Octopuses and Dolphins, she presents it to them as a quite novel method, saying "I'll show you a different way to do it ... you start with the lower number ...". Naomi makes no attempt to connect it to what they were doing (or supposed to be doing) earlier in the lesson. Again, the link might be prompted by a different choice of examples. Naomi talks about putting 8 cubes in a row, then 10, so that "Then you can *see* what the difference is". This *see*ing can be literally subitising, as opposed to seeing as 'finding out'. If one lined up, say, 4 cubes and 13 cubes, it would be harder to subitise the difference, and a link with counting would be more natural and expedient.

Concerning connection, we might identify for later discussion with Naomi matters such as:

- She has now taught two structures for subtraction (take away and difference), each with one or more associated procedures. How might she help the children to reconcile the two? How might they be modelled on an empty number line? Are there other structures that they might meet in the future?

Contingency

Naomi does not invite or explore the children's own proposals for the solution of the comparison problems, or probe the ways that they are making sense of the lesson. There were times when children offered her an opportunity to do so. For example, in the Plenary, Gavin offered a useful insight after (as we saw earlier) Jim was about to add 6 to 3. We pick up the story at the point where we left it.

Naomi:	Not adding, it's the difference ... Gavin, do you want to help us out.
Gavin:	If you wanted to make six, you could add three there, and make another three, and make six.

Naomi:	So three and three makes six. So how can we say that with the difference. What's the difference between three and six? What's the difference between the two numbers? How many more?
Gavin:	If you wanted to add on three to make six you'd make six, but if you wanted to take away three to make three, it'll be three.
Naomi:	OK, I know what you mean Gavin. Good, so the difference is three. We can say the difference between three and six is three. If we start with three ... Madeleine?
Madeleine:	It's one.
Naomi:	I'm not sure what you mean. The difference between three and six. [Counts on her fingers] Three, four, five, six.
Madeleine:	The answer's one.
Naomi:	So it's three.

Gavin seems to be on to something. He can answer Naomi's difference question, not by counting up, but by performing an addition! Naomi claims to know what Gavin means, but makes no capital from the connection that he offers, which relates comparison to addition, and then addition to "take away". Madeleine, on the other hand, is one of the many children still experiencing significant problems. Naomi is "not sure" what she means, and takes no steps to find out. *Why* does Madeleine insist that the answer is one? How does she 'construct' that answer? If only Naomi could find out, she might stand a better chance of getting to the bottom of the incomprehension of so many of the class.

At the beginning of the Plenary, the dice show 3 and 5, and, as we saw earlier, Jeffrey sums them and answers 8. Stuart then comes to the rescue with 2.

Naomi:	Excellent. Can you tell us how you worked it out. Come to the front. Olive, stand up! Sit in your rows please. Right, Stuart just worked out the difference between three and five and said it was two. How did you work it out, Stuart?
Stuart:	I held out three fingers and five, and then there's two left.

Whereas Naomi had used her fingers as a way of tallying when counting-on, Stuart has used his to model the matching procedure. He is using his fingers as portable manipulatives, representing *both* sets simultaneously – as Naomi had shown them to do with the frogs at the beginning of the lesson. Naomi responds:

Naomi:	Ah, OK. That does work because you've got five fingers on your hands so if you've got five here and three you've got two left to make five. But I know an even better way to work it out. Does anybody know another way to work it out?

Naomi seems not to have seen the significance of Stuart's unexpected explanation, and persists ("But I know an even better way") with urging them to count on from the smaller number.

Finally, we note an interesting intervention by Hugh very early in Naomi's Introduction to the Main Activity, when she had established that she had two more frogs than her neighbour.

Hugh:	You could both have three, if you give one to your neighbour.
Naomi:	I could, that's a very good point, Hugh. I'm not going to do that today though. I'm just going to talk about the difference. Madeleine, if you had a pond, how many frogs would you like in it?

One can readily sympathise with Naomi's response to Hugh's insight, which seems to deviate too far from the agenda that she had set for the lesson. Such moments, when the lesson is poised to be hijacked, often occur in mathematics teaching. Naomi acknowledges Hugh's observation, but refuses to be diverted from her course. With the benefit of hindsight, one can see that she had the option, if she were brave (or confident, or reckless) enough to choose it, to take Hugh's remark as the starting point of rather a nice enquiry. If I have n things and you have m, what must be done so that we have the same number? Can it always be done? This would almost certainly have prompted investigation of the difference between two numbers, as well as halving, and the distinction between even and odd numbers. Of course, it is relatively easy for mathematics educators to say such things from the comfort of their office chair, as we acknowledge in the discussion to follow.

Finally then, if the review discussion were to include contingency, one might raise for discussion with Naomi matters such as:

- Did she recall Hugh's comment in the Introduction to the Main Activity? What considerations led her to praise him but not to follow it up?
- What might be a good way to respond to answers such as Madeleine's, that seem to bear little relation to the problem under consideration?
- Did she notice that Stuart's explanation, in the Plenary, referred back to the matching strategy that she had taught the class initially?

CONCLUSIONS AND CAVEATS

In this paper, we have introduced and elucidated the knowledge quartet – foundation, transformation, connection and contingency – and used it to raise issues related to Naomi's knowledge – her PCK especially – in our analysis of her lesson on subtraction with a Year 1 class.

We set out to develop an empirically based conceptual framework for the discussion of mathematics content knowledge, between teacher educators, trainees and teacher-mentors, in the context of school-based placements. The outcome has been the knowledge quartet. This framework is entering an extended period of evaluation with our students and colleagues in the current academic year and beyond. We have a manageable framework within which to discuss actual, observed teaching sessions with trainees and their mentors. These groups of participants in ITT, as well as our university-based school 'partnership tutor' colleagues, are becoming acquainted with (and convinced of the value of) the quartet, and becoming familiar with some details of its conceptualisation, as described in this paper. Early indications are that it is being well received by mentors, who like the specific focus on *mathematics* content and pedagogy. They observe that it compares favourably with guidance on mathematics lesson observation from the NNS itself, which focuses on more generic issues such as "a crisp start, a well-planned middle and a rounded end. Time is used well. The teacher keeps up a suitable pace and spends very little time on class organisation, administration and control." (DfEE, 2000, p. 11).

Placement lesson observation is normally followed by a review meeting between partnership tutor (and/or mentor) and trainee. The quartet offers a workable framework for reflection on mathematics subject matter issues in the review meeting. Research shows that such meetings typically focus heavily on organisational features of the lesson, with very little attention to mathematical aspects of mathematics lessons (Brown, McNamara, Jones & Hanley, 1999). In one recent study, only 2% of mentors' suggestions to beginning teachers related to the subject matter being taught (Strong & Baron, 2004). The availability of the quartet might encourage and assist greater attention to subject matter content in the review. Indications of how this might work are implicit or explicit in our analysis of Naomi's lesson. In particular, we are aware that our analysis has been *selective*: we raised for attention some issues, but there were others which, not least out of space considerations, we chose not to mention. The same would be likely to be true of the review meeting – in that case due to *time* constraints, but also to avoid overloading the trainee with action points. Each such meeting might well focus on only one or two dimensions of the knowledge quartet for similar reasons.

It is relevant and important to return here to our earlier comment about the futility of stating what Naomi *ought* to know at this stage of her teacher education, or what she ought to have done in the lesson. Maintaining a commitment to development rather than to judgement in the application of the 'quartet' is likely to be a challenge for teacher educators. What we can say, as university tutors, is that Naomi had met all the 'academic' demands of the course – essays and audits of various kinds – and now she was responding to a different challenge, and with the same intelligent determination that she had applied to other course requirements. The acts of lesson preparation and teaching call on all kinds of personal and intellectual qualities, and the trainee teacher's performance in the classroom is shaped by a host of factors in addition to SMK and PCK. In the novice teacher we see the very beginnings of a process of reconciliation of pre-existing beliefs, new 'theoretical' knowledge, 'practical' advice received from various quarters, in the context of highly-pressured, high-stakes school-based placements. Here, trainees begin to perceive some disparity between "college ideals" and the practice that they witness in schools (Eisenhart et al., 1993; Smith, 1999). Understandably, their concern for pupil learning is often eclipsed by anxieties about class management and control of behaviour (Brown et al., 1999). This sense of priority is well-judged: regardless of their subject matter knowledge base, prospective teachers who do not establish classroom norms and routines for discipline, management, and instruction are often unable to focus on what students are learning (Hollingsworth, 1989). Indeed, it is apparent from many of Naomi's utterances quoted earlier (such as "Jeffrey, come and sit here. Hugh, you can sit down.") that interruptions and disturbances of various kinds were a constant distraction and a concern to her throughout her lesson. This impression is confirmed by the researcher who observed and videotaped the lesson. This would go a long way towards accounting for Naomi's reluctance to explore children's ideas if they seemed to deviate from her chosen agenda.

In conclusion, the knowledge quartet is sufficient as a 'tool' with which to home in on subject matter in lesson observation review, and is promising as a framework to develop teaching and teacher knowledge. At the same time, sensitive consideration of subject matter issues must necessarily take into account the anxieties experienced by novice teachers, and the tensions and constraints identified by Eisenhart et al. (1993).

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NOTES

[1] The official discourse in England refers to students undergoing pre-service preparation for school teaching as 'trainees'. In this paper, we speak of 'students', 'prospective teachers' and 'trainees' synonymously.

[2] We use 'primary' and 'elementary' synonymously in this paper to refer to the phase of schooling for ages 3/4 to 11.

[3] The examination for the General Certificate in Secondary Education (GCSE) is normally taken at age 16, after which further study of mathematics is optional within the English education system.

[4] Students normally study three or four subjects for the Advanced Level General Certificate in Education (A-level GCE) between the ages of 16 and 18. Grades A to E are pass grades, A being the highest. Naomi had opted for English Language, Mathematics and French, achieving grades A, B, B respectively. She also undertook a one-year 'Advanced Supplementary' course in Psychology and achieved a grade B.

[5] Compulsory education in England and Wales is organised in chronological 'Years', normally beginning at age four or five with between one and three terms in Year R (for 'reception'). The youngest children in Year 1 will be just five at the beginning of the academic year, the oldest nearly seven at the end.

[6] The National Numeracy Strategy *Framework* (DfEE, 1999) guidance segments each mathematics lesson into three distinctive and readily-identifiable phases: the *mental and oral starter*; the *main activity* (an *introduction* by the teacher, followed by *group work*, with tasks differentiated by pupil attainment); and the concluding *plenary*.

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